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2nd Order Accuracy: MUSCL Scheme

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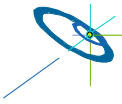

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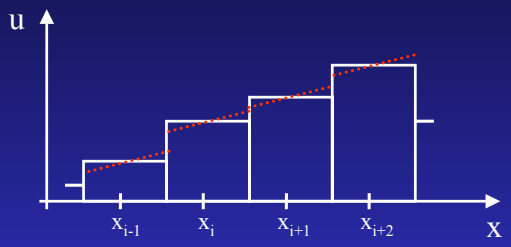
9. Second Order Accuracy- MUSCL Schemes

MUSCL-Idea (van Leer, 1978)



Slope calculation: TVD-, ENO-, WENO-Method

Scalar theory extended to systems

Piecewise linear approximation

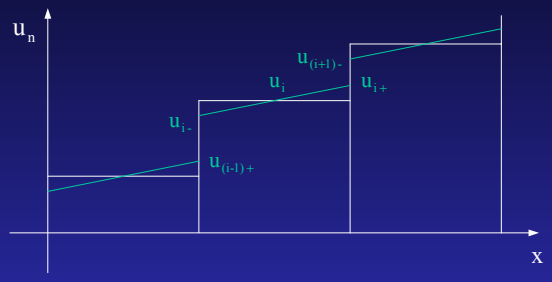
$$u^n(x) = u_i^n + (x - x_i)s_i^n \quad , \quad x \in [x_{i-1/2}\,, x_{i+1/2}]$$

s_i^n slope in I_i

Second order accuracy in space

$$u(x\,, t_n) = u(x_i\,, t_n) + (x - x_i)u_x(x\,, t_n) + O\left((x - x_i)^2\right)$$

$$\Rightarrow \quad \boxed{s_i^n = u_x(x_i\,, t_n) + O\left((x - x_i)^2\right)}$$



Reconstruction of a piecewise linear approximation

Taylor expansion in time

$$u\left(x\,,\,t+\tfrac{\Delta t}{2}\right)=u\left(x\,,\,t\right)+\frac{\Delta t}{2}u_t\left(x\,,\,t\right)+O\left(\Delta t^2\right)$$

$$u_t=-f\left(u\right)_x$$

$$u\left(x\,,\,t+\tfrac{\Delta t}{2}\right)=u\left(x\,,\,t\right)-\frac{\Delta t}{2}f\left(u\left(x\,,\,t\right)\right)_x+O\left(\Delta t^2\right)$$

$$\Longrightarrow\quad u_{i\pm}^{n+\frac{1}{2}}=u_{i\pm}^{n}-\frac{\Delta t}{\Delta x}\left(f\left(u_{i+}^{n}\right)-f\left(u_{i-}^{n}\right)\right)$$

MUSCL - Procedure

Boundary v alues at t_n

$$u_{i\pm}^{n} = u_i^n \pm \frac{\Delta x}{2} \cdot s_i^n$$

$t_n \rightarrow t_{n+1/2}$

$$u_{i\pm}^{n+1/2} = u_{i\pm}^{n} - \frac{\Delta t}{2 \cdot \Delta x}\left(f\left(u_{i+}^{n}\right) - f\left(u_{i-}^{n}\right)\right)$$

FV - scheme

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(g_{i+1/2}^{n+1/2} - g_{i-1/2}^{n+1/2}\right)$$

with $\quad g_{i+1/2}^{n+1/2} = g\left(u_{i+}^{n+1/2}, u_{(i+1)-}^{n+1/2}\right)$

Slope Calculation

Mathematical theory for scalar convention laws

TVD-property (Total Variation Diminishing)

$$\sum_{all\ i}\left|u_{i+1}^{n} - u_i^n\right| \le \sum_{all\ i}\left|u_{i+1}^{0} - u_i^0\right|$$

Sufficient condition (A. Harten)

$$0 \le \left\{\frac{\Delta x\, s_i}{u_i - u_{i-1}}, \frac{\Delta x\, s_i}{u_{i+1} - u_i}\right\} \le 2$$

Examples

1. Minmod-function

$$s_i = \frac{1}{\Delta x}\,\text{min mod}\left(u_{i+1}-u_i\,,\,u_i-u_{i-1}\right)$$

$$\text{min mod}\left(a\,,\,b\right)=\begin{cases} a & \text{for} \quad |a|<|b|\,,\; ab>0 \\ b & \text{for} \quad |a|\geq|b|\,,\; ab>0 \\ 0 & \text{otherwise} \end{cases}$$

2. Sweby's slope calculation

$$s_k\left(a,b\right)=\text{sign}\,a\,\max\left\{\left|\text{minmod}\left(a\,,\,kb\right)\right|\,,\left|\text{minmod}\left(ka\,,\,b\right)\right|\right\}$$

$$\text{with} \qquad 1\leq k\leq 2$$

Extension to Euler equations

1. Calculate slopes in primitive variables: ρ, υ, p

2. Calculate slopes in characteristic variables e. g.

$$s_i^n = \sum_{j=1}^{3}\text{min mod}\left(\alpha_j\,,\,\beta_j\right)r_j$$

with

$$\sum_{j=1}^{3}\alpha_j\,r_j = \frac{1}{\Delta x}\left(u_{i+1}-u_i\right)$$

$$\sum_{j=1}^{3}\beta_j\,r_j = \frac{1}{\Delta x}\left(u_i-u_{i-1}\right)$$